Models of Set Theory II - Winter 2017/2018

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Problem 1 (6 points)**.** Let M be a transitive and countable model of ZFC and let $\mathbb{P} \in M$ be an $\aleph_1$-closed notion of forcing. Suppose that T is an ω_1-tree in M such that every level of T is countable. Prove that T has no new branches in $M[G]$.

Hint: Suppose for a contradiction that T has a branch $b \in M[G]$ which is not in M (note that b has to have length ω_1). Then there is a name $\dot{b}$ for b and a condition $p_\emptyset \in G$ such that $p_\emptyset \Vdash \dot{b} \neq \check{a}$ for all $a \in M$. Construct by induction $p_s < p_\emptyset$ and nodes $x_s \in T$ for all finite sequences s of 0's and 1's in a way that $p_{s^\frown 0}$ and $p_{s^\frown 1}$ are both stronger than p_s, $p_{s^\frown 0} \Vdash \check{x}_{s^\frown 0} \in \dot{b}$ and $p_{s^\frown 1} \Vdash \check{x}_{s^\frown 1} \in \dot{b}$, where $x_{s^\frown 0}$ and $x_{s^\frown 1}$ are at the same level of T (for p_s constructed, we can find two incomparable nodes $x_{s^\frown 0}$ and $x_{s^\frown 1}$ such that they are both $>_T x_s$ and that they are at the same level of T, and $p_{s^\frown 0}$ and $p_{s^\frown 1}$ both stronger than p_s such that $p_{s^\frown 0} \Vdash \check{x}_{s^\frown 0} \in \dot{b}$ and $p_{s^\frown 1} \Vdash \check{x}_{s^\frown 1} \in \dot{b}$). Then there is a level $\alpha < \omega_1$ such that all x_s lie below that level. Show that the αth level of T has at least $2^{\aleph_0}$ elements (using that $\mathbb{P}$ is an $\aleph_1$-closed notion of forcing and that $p_\emptyset$ forces that $\dot{b}$ is uncountable), contrary to our assumption.

Problem 2 (2 points)**.** Let κ be an infinite cardinal and $\mathbb{P}$ be a κ-closed partial order. Show that $\mathrm{FA}_\kappa(\mathbb{P})$ holds.

Problem 3 (2 points)**.** Prove that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{Fn}(\aleph_0, \aleph_0, \aleph_0))$ is false.

Problem 4 (4 points)**.** Suppose that M is a transitive and countable model of ZFC. Let $\mathbb{P} \in M$ be a forcing notion and let G be M-generic for $\mathbb{P}$. Let $[\omega]^\omega$ denote the set of infinite subsets of ω. For $x, y \in [\omega]^\omega$ we say that x *splits* y if both $y \cap x$ and $y \setminus x$ are in $[\omega]^\omega$. A set $x \in ([\omega]^\omega)^{M[G]}$ is said to be a *splitting real* if it splits every set in $([\omega]^\omega)^M$. Let $\mathbb{C}$ denote Cohen forcing. Prove that $\mathbb{C}$ adds splitting reals.

Problem 5 (6 points)**.** Let M be a transitive and countable model of ZFC + GCH and let G be M-generic for $\mathrm{Fn}(\aleph_2^M \times \omega, 2, \aleph_0)$. Prove that in $M[G]$ we have that $\mathrm{non}(\mathcal{N}) = \aleph_2$ and $\mathrm{add}(\mathcal{N}) = \aleph_1$.

Please hand in your solutions on Monday, October 23 before the lecture.